Properties of a New Generalized Family of Distributions with Application to Relief Times of Patients Data

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Abstract

Statisticians have developed and proposed new families of distributions by extending or generalizing the existing ones. These families of distributions are extended by adding one or more parameters to the baseline distributions to make it more flexible in fitting different kinds of data. In this study, a new four-parameter lifetime family of distributions called the Extended Topp Leone Generalized Exponentiated-G family of distributions was introduced by using the T-X methodology which has been proposed in the literature. Some mathematical properties of the class of distributions such as the moments, moment generating function, quantile function, survival, hazard, reversed hazard and odds functions and Renyi entropy were presented. The estimation of the parameters by maximum likelihood method was discussed. Two sub-models such as extended Topp Leone generalized exponentiated exponential and extended Topp Leone generalized exponentiated Weibull distributions were discussed. An application of extended Topp Leone generalized exponentiated exponential distribution to a real life data set representing the relief times (in minutes) of patients receiving an analgesic was used to show the fit and flexibility of the new distribution over some lifetime distributions in literature. The results showed that the new distribution fits better in this data than the competing models considered.

Keywords: Exponentiated Kumaraswamy; Odd Lindley-G; Generalized-G Family; Flexibility; Renyi Entropy

1. Introduction

In recent times, the interest of researchers has been on defining new generators or generalizing families of continuous distributions by introducing additional shape parameter(s) to a baseline distribution. The addition of parameters has been proved useful in exploring skewness and tail properties, and also for improving the goodness-of-fit of the generated family. Some of the recent families of distributions appearing in literature are: The Kumaraswamy-G family of distributions by Cordeiro and de Castro (2011), Topp–Leone exponentiated generalized-G class of distributions by Rasheed (2020), Exponentiated Weibul-H family of distributions by Cordeiro et al., (2017), The Odd Lindley-G family of distributions by Gomes-Silva (2017), The Topp Leone odd Lindley-G family of distributions by Reyad et al., (2018), Odd Frechet-G family of distributions by Haq and Elgarhy (2018), Topp-Leone generated family of distributions by Aryuyuen (2018), The Marshall-Olkin Odd Lindley-G Family of Distributions by Jamal (2019), The Exponentiated Kumaraswamy-G family of distributions by Silva et al., (2019), Fréchet Topp Leone G Family of Distributions by Reyad et al., (2019), Power Lindley-G Family of Distributions by Hassan and Nassr (2019), Modi family of continuous probability distributions by Modi et al., (2020), The Topp Leone Exponentiated-G family of distributions by Ibrahim et al., (2020a), The Topp Leone Kumaraswamy-G family of distributions by Ibrahim et al., (2020b), Odd Chen-G family of distributions Anzagra et al., (2020).

2. Materials and Methods

Let r(t) be the probability density function (pdf) of a random variable $T \in [a,b]$ for $-\infty \le a < b < \infty$ and let W[G(x)] be a function of the cumulative distribution function (cdf) of a random variable X such that W[G(x)] satisfies the following conditions:

(i) $W[G(x)] \in [a,b]$

(ii) $W[G(x)]$ is differentiable and monotonically non-decreasing, and (1)

(iii) $W[G(x)] \to a$ a as $x \to \infty$ and $W[G(x)] \to b$ as $x \to \infty$

Then, Alzaatreh et al., (2013) defined the T-X family of distributions by

$$F(x) = \int_a^{W[G(x)]} r(t)dt \,. \tag{2}$$

Where $W[G(x)]$ satisfies (1). The pdf corresponding to (2) is given by

$$f(x) = \left\{ \frac{dW[G(x)]}{dx} \right\} r(W[G(x)]) \,. \tag{3}$$

If $H(x;\xi)$ denotes the cumulative distribution function (cdf) of a random variable x, with corresponding pdf $h(x;\xi)$, then the Exponentiated-G family is given as

$$G(x;\alpha,\xi) = \left[H(x;\xi) \right]^{\alpha} \tag{4}$$

and pdf corresponding to (4) is

$$g(x;\alpha,\xi) = \alpha h(x;\xi) H(x;\xi)^{\alpha-1} \tag{5}$$

Let $g(x;\alpha,\xi)$ and $G(x;\alpha,\xi)$ denote the pdf and cdf of a baseline distribution with parameter vector ξ, then Rasheed (2020) defined the Topp-Leone-Exponentiated generalized-G class of distributions with cdf given as

$$F(x;\theta,\lambda,b,\xi) = \left\{ 1 - \left[1 - \left[1 - \left[1 - G(x;\xi) \right]^{\lambda} \right]^{b} \right]^{2} \right\}^{\theta} \tag{6}$$

and pdf corresponding to (6) is

$$f(x;\theta,\lambda,b,\xi)=2\alpha b\theta\lambda g(x;\xi)[1-G(x;\xi)]^{\alpha-1}[1-[1-G(x;\xi)]^{\alpha}]^{b-1} \times[1-[1-[1-G(x;\xi)]^{\alpha}]^{b}]\{1-[1-[1-[1-G(x;\xi)]^{\lambda}]^{b}]^{2}\}^{\theta-1} \quad (7)$$

2.1. The extended Topp Leone exponentiated Generalized-G Family of Distributions

Based on the idea of T-X family pioneered by Alzaatreh et al., (2013), we introduced a new family of continuous distributions called the Extended Topp Leone Exponentiated generalized-G (ETLExGen-G) family of distributions with cdf given by

$$F\left(x;\alpha,\lambda,\theta,b,\xi\right)=\int_{0}^{\left[G(x;\alpha)\right]^{\alpha}}2ab\theta\lambda g\left(t;\xi\right)\left[1-G\left(t;\xi\right)\right]^{\lambda-1}\left[1-\left[1-G\left(t;\xi\right)\right]^{\lambda}\right]^{b-1} \times\left[1-\left[1-\left[1-G(t;\xi)\right]^{\lambda}\right]^{b}\right]\left\{1-\left[1-\left[1-\left[1-G(t;\xi)\right]^{\lambda}\right]^{b}\right]^{2}\right\}^{\theta-1}dt \quad (8)$$

where $G(t;\xi)$ is given in (4) and $g(t;\xi)$ is defined in (5).

Let $\varpi=1-\left[1-\left[1-\left[1-G\left(t;\xi\right)\right]^{\lambda}\right]^{b}\right]^{2}$,

$$d\varpi=2ab\lambda\left[1-G\left(t;\xi\right)\right]^{\lambda-1}\left[1-\left[1-G\left(t;\xi\right)\right]^{\lambda}\right]^{b-1}\left[1-\left[1-\left[1-G(t;\xi)\right]^{\lambda}\right]^{b}\right]g(t;\xi)dt$$

İf $t=0,\varpi_0=0$ and if $t=x,\varpi_x=1-\left[1-\left[1-\left[1-G\left(x;\xi\right)\right]^{\lambda}\right]^{b}\right]^{2}$

$$F\left(x;\alpha,\lambda,\theta,b,\xi\right)=\int_{0}^{\varpi_x}\theta\varpi^{\theta-1}dy$$

$$F\left(x;\alpha,\lambda,\theta,b,\xi\right)=\left.\frac{\theta\varpi^{\theta}}{\theta}\right|_{0}^{\varpi_x}$$

$$F\left(x;\alpha,\lambda,\theta,\xi\right)=\left.\varpi^{\theta}\right|_{0}^{\varpi_x}$$

$$F\left(x;\alpha,\lambda,\theta,b,\xi\right)=\left[1-\left[1-\left[1-\left[1-G\left(x;\xi\right)\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta}$$

but from (4),

$$G\left(x;\alpha,\xi\right)=\left[H\left(x;\xi\right)\right]^{\alpha}$$

Therefore,

$$F\left(x;\alpha,\lambda,\theta,b,\xi\right)=\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta} \quad (9)$$

Where $H\left(x;\xi\right)$ is the cdf of the baseline distribution with parameter vector ξ. The corresponding pdf to (9) is

$$f\left(x;\alpha,\lambda,\theta,b,\xi\right)=2\alpha\lambda\theta bh\left(x;\xi\right)\left[H\left(x;\xi\right)\right]^{\alpha-1}\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda-1}\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b-1}$$
$$\times\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta-1} \quad (10)$$

which is the pdf of the proposed ETLExGen-G family of distributions. Also, $x\geq 0$, $\alpha,\lambda,\theta,b,\xi>0$. This new family of distribution is proposed to improve the fitness and vary tail weight of the existing distributions in fitting real life data sets that exhibit different shapes.

2.2. Infinite mixture representations

Here, the infinite mixture representations for the cdf and pdf of the ETLExGen-G family are given in terms of baseline densities. Consider following series expansion

$$\left(1-y\right)^{z}=\sum_{i=0}^{\infty}\binom{z}{i}\left(-1\right)^{i}y^{i}$$

Using (9), we have the expansion for the density as

$$\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta-1}=\sum_{i=0}^{\infty}\binom{\theta}{i}\left(-1\right)^{i}\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2i}$$

$$\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2i}=\sum_{j=0}^{\infty}\binom{2i}{j}\left(-1\right)^{j}\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{bj}$$

$$\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{bj}=\sum_{k=0}^{\infty}\binom{bj}{k}\left(-1\right)^{k}\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda k}$$

$$\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda k}=\sum_{l=0}^{\infty}\binom{\lambda k}{l}\left(-1\right)^{l}H\left(x;\xi\right)^{\alpha l}$$

$$H\left(x;\xi\right)^{\alpha l}=\sum_{p=0}^{\infty}\sum_{q=l}^{\infty}\binom{\alpha l}{l}\binom{l}{p}\left(-1\right)^{p+l}\left[G(x;\xi)\right]^{p}$$

$$F\left(x;\alpha,\lambda,\theta,b,\xi\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\theta}{i}\binom{2i}{j}\binom{bj}{k}\binom{\lambda k}{l}\left(-1\right)^{i+j+k+l}\sum_{p=0}^{\infty}\sum_{p=l}^{\infty}\binom{\alpha l}{l}\binom{l}{p}\left(-1\right)^{p+l}\left[H\left(x;\xi\right)\right]^{p}$$

$$F\left(x;\alpha,\lambda,\theta,b,\xi\right)=\sum_{p=o}^{\infty}\Delta_{p}G_{p}(x;\xi) \quad (11)$$

Where

$$\Delta_{p}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\theta}{i}\binom{2i}{j}\binom{bj}{k}\binom{\lambda k}{l}\left(-1\right)^{i+j+k+l}\sum_{p=l}^{\infty}\binom{\alpha l}{l}\binom{l}{p}\left(-1\right)^{p+l}$$

$$\left[H\left(x;\xi\right)\right]^{p}=G_{p}\left(x;\xi\right).$$

Similarly, we have an expansion for the pdf of ETLExGen-G family as

$$f\left(x;\alpha,\lambda,\theta,b,\xi\right)=\sum_{p=o}^{\infty}\Delta_{p}g_{p-1}(x;\xi), \tag{12}$$

where $g_{p-1}\left(x;\xi\right)=pg\left(x;\xi\right)G\left(x;\xi\right)^{p-1}$.

And $G_p(x)$ denotes the Exp-G cdf with power parameter p. Equation (12) shows that the ETLExGen-G density function is a linear combination of Exp-G densities. Thus, some structural properties of the ETLExGen-G class such as the ordinary and incomplete moments and moment generating function can be obtained from well-known Exp-G properties.

2.2 Properties

In this subsection, we derived some of the mathematical properties of the new family of distributions.

2.2.1 Moments

$$\mu_{r}^{'}=E\left(x^{r}\right)=\int_{0}^{\infty}x^{r}f\left(x\right)dx \tag{13}$$

Using the infinite mixture representation of the pdf in (11), we have

$$E\left(x^{r}\right)=\sum_{p=0}^{\infty}\Delta_{p}\int_{0}^{\infty}x^{r}g_{p-1}\left(x\right)dx \tag{14}$$

2.2.2 Incomplete moments

Similarly, the s^{th} incomplete moment is given as

$$E\left(x^{r}\right)=\int_{0}^{x}x^{s}f\left(x\right)dx \tag{15}$$

The s^{th} incomplete moment of ETLExGen-G family is given as

$$E\left(x^{s}\right)=\sum_{p=0}^{\infty}\Delta_{p}\int_{0}^{x}x^{s}g_{p-1}\left(x\right)dx \tag{16}$$

2.2.3 Moment generating function

$$E\left(e^{tx}\right)=\int_{0}^{\infty}e^{tx}f\left(x\right)dx \tag{17}$$

The moment generating function of ETLExGen-G family is given as

$$E\left(e^{tx}\right)=\sum_{p=0}^{\infty}\Delta_p\int_0^{\infty}e^{tx}g_{p-1}\left(x\right)dx \tag{18}$$

Note that the integrals depend on the choice of the baseline distribution.

2.2.4 Renyi entropy

The deviation of the uncertainty is measured by the entropy of a random variable which is defined by Rényi (1961). A large value of the entropy indicates the greater uncertainty in the data. The Rényi entropy is defined by (for $R>0$ and for $R\neq 1$)

$$I_R=\frac{1}{1-R}log\int_0^{\infty}f^R\left(x\right)dx \tag{19}$$

$$f^R\left(x\right)=\left[\begin{array}{c}2\alpha\lambda\theta bh\left(x;\xi\right)\left[H\left(x;\xi\right)\right]^{\alpha-1}\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda-1}\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b-1}\\ \times\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta-1}\end{array}\right]^R$$

$$f^R\left(x\right)=\left[2\alpha\lambda\theta b\right]^R\left[h(x;\xi)\right]^R\left[H(x;\xi)\right]^{R(\alpha-1)}\left[1-H\left(x;\xi\right)^{\alpha}\right]^{R(\lambda-1)}\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{R(b-1)}$$
$$\times\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^R\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{R(\theta-1)}$$

$$f^R\left(x\right)=\left[2\alpha b\lambda\theta\right]^R\left[h\left(x;\xi\right)\right]^R\left[H\left(x;\xi\right)\right]^{R(\alpha-1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{i+j+k+l}\binom{R\left(\theta-1\right)}{i}\binom{2i+R}{j}\binom{b\left(R+j\right)-R}{k}\binom{\lambda\left(k+R\right)-R}{l} \tag{20}$$

$$f^R\left(x\right)=\sum_{i=0}^{\infty}V_i\left(l\right)h_{\alpha(l+R)-R}\left(x;\xi\right) \tag{21}$$

where,

$$V_i\left(l\right)=\left(2\alpha\lambda b\theta\right)^R\sum_{j=0}^{\infty}\binom{R\left(\theta-1\right)}{i}\binom{2i+R}{j}\binom{b\left(R+j\right)-R}{k}\binom{\lambda\left(k+R\right)-R}{l}\frac{\left(-1\right)^{i+j+k+l}}{\alpha\left(l+R\right)-R+1}$$

$$h_{\alpha\left(l+R\right)-R}\left(x;\xi\right)=\left(\alpha\left(l+R\right)-R+1\right)h\left(x;\xi\right)^R H\left(x;\xi\right)^{\alpha\left(l+R\right)-R}.$$

Then, the Renyi entropy of ETLExGen-G family is given as

$$I_R=\frac{1}{1-R}log\left(\sum_{i=0}^{\infty}V_i\left(l\right)\right)\int_0^{\infty}h_{\alpha\left(l+R\right)-R}\left(x;\xi\right)dx \tag{22}$$

2.2.5 Survival function

The survival function, which is the probability of an item not failing prior to some time, can be defined as

$$S(x;\alpha,\theta,\lambda,b,\xi)=1-F(x;\alpha,\theta,\lambda,b,\xi) \tag{23}$$

The survival function of the ETLExGen-G family of distributions is given as

$$S(x;\alpha,\theta,\lambda,b,\xi)=1-\left[1-\left[1-\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta} \tag{24}$$

2.2.6 Hazard rate function

The hazard rate function is an important measure use to characterize a life phenomenon. It is given as

$$\tau(x;\alpha,\theta,\lambda,b,\xi)=\frac{f(x;\alpha,\theta,\lambda,b,\xi)}{S(x;\alpha,\theta,\lambda,b,\xi)} \tag{25}$$

The hazard rate function of the ETLExGen-G family of distributions is given as

$$\tau(x;\alpha,\theta,\lambda,b,\xi)=\frac{2\alpha\lambda\theta bh(x;\xi)\left[H(x;\xi)\right]^{\alpha-1}\left[1-H(x;\xi)^{\alpha}\right]^{\lambda-1}\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b-1}\left[1-\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b}\right]\left[1-\left[1-\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta-1}}{1-\left[1-\left[1-\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta}} \tag{26}$$

2.2.7 Reversed hazard rate function

The reverse hazard rate function is defined as

$$\varnothing(x;\alpha,\theta,\lambda,b,\xi)=\frac{f(x;\alpha,\theta,\lambda,b,\xi)}{F(x;\alpha,\theta,\lambda,b,\xi)} \tag{27}$$

Then, the reverse hazard rate function of the ETLExGen-G family of distributions is given as

$$\varnothing(x;\alpha,\theta,\lambda,b,\xi)=\frac{2\alpha\lambda\theta bh(x;\xi)\left[H(x;\xi)\right]^{\alpha-1}\left[1-H(x;\xi)^{\alpha}\right]^{\lambda-1}\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b-1}\left[1-\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b}\right]\left[1-\left[1-\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta-1}}{\left[1-\left[1-\left[1-\left[1-H(x;\xi)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta}} \tag{28}$$

2.2.8 Odds Function

$$O(x;\alpha,\theta,\lambda,b,\xi)=\frac{F(x;\alpha,\theta,\lambda,b,\xi)}{S(x;\alpha,\theta,\lambda,b,\xi)} \tag{29}$$

Then, the odds function of the ETLExGen-G family of distributions is given as

$$O\left(x;\alpha,\theta,\lambda,b,\xi\right)=\frac{\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta}}{1-\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta}} \tag{30}$$

2.2.9 Cumulative Hazard Function

$$C\left(x;\alpha,\theta,\lambda,b,\xi\right)=-\ln\left(S\left(x;\xi\right)\right) \tag{31}$$

Then, the cumulative hazard function of the ETLExGen-G family of distributions is given as

$$C\left(x;\alpha,\theta,\lambda,b,\xi\right)=-\ln\left[1-\left[1-\left[1-\left[1-\left[1-H\left(x;\xi\right)^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]^{\theta}\right] \tag{32}$$

2.2.10 Quantile Function

The ETLExGen-G family is easily simulated by inverting (9) as follows: if u has a uniform U(0,1) distribution, then the solution of the nonlinear equation is given by

$$x=Q\left(u\right)=H^{-1}\left\{\left[1-\left[1-\left[1-\left[1-U^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right]^{\frac{1}{b}}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right\} \tag{33}$$

Where H^{-1} is the quantile function of the baseline distribution, $H(x;\xi)$

In particular, the median of the ETLExGen-G family of distributions can be derived by substituting u = 0.5 into (33) as follows:

$$Q\left(0.5\right)=H^{-1}\left\{\left[1-\left[1-\left[1-\left[1-0.5^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right]^{\frac{1}{b}}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right\} \tag{34}$$

2.3 Distribution of order statistics

Let $X_1,X_2,X_3,\ldots,X_n$ be a random sample and its ordered values are denoted as $X_{(1)},X_{(2)},X_{(3)},\ldots,X_{(n)}$. The pdf of order statistics is obtained using the below function

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} \tag{35}$$

2.3.1 Minimum order statistics

The minimum order statistics is obtained by setting $r=1$ in (35) as

$$f_{1:n}(x)=nf(x)\left[1-F(x)\right]^{n-1} \tag{36}$$

Then, the minimum order statistics of the ETLExGen-G family of distributions is given as

$$f_{1:n}(x)=2n\alpha b\theta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+k+l}\,\Gamma(n)\Gamma(\theta(i+1))\Gamma(2(j+1))\Gamma(b(k+1))\Gamma(\lambda(l+1))}{i!\,j!\,k!\,l!\,\Gamma(n-i)\Gamma(\theta(i+1)-j)\Gamma(2(j+1)-k)\Gamma(b(k+1)-l)\Gamma(\lambda(l+1)-m)}h(x)\left[H(x)\right]^{\alpha(m+1)-1} \tag{37}$$

2.3.2. Maximum order statistics

The maximum order statistics is obtained by setting $r=n$ in (35) as

$$f_{n:n}(x)=nf(x)\left[F(x)\right]^{n-1} \tag{38}$$

Then, the maximum order statistic of the ETLExGen-G family of distributions is given as

$$f_{n:n}(x)=2n\alpha b\theta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\,\Gamma(\theta n)\Gamma(2(i+1))\Gamma(b(j+1))\Gamma(\lambda(k+1))}{i!\,j!\,\Gamma(\theta n-i)\Gamma(2(i+1)-j)\Gamma(b(j+1)-k)\Gamma(\lambda(k+1)-l)}h(x;\varphi)H(x;\varphi)^{\alpha(l+1)-1} \tag{39}$$

2.4 Maximum Likelihood Estimation

Let $X_{1,}X_2,\ldots,X_n$ be an iid observed random sample of size n from the ETLExGen-G family. Then, the log-likelihood function based on observed sample for the vector of parameter $\varnothing=(\alpha,\theta,\lambda,b)^T$ is given by

$$\begin{aligned}L(\varnothing)&=nlog2+nlog\alpha+nlog\theta+nlog\lambda+nlogb+\sum_{i=1}^{n}\log\left[h(x_i;\xi)\right]+(b-1)\sum_{i=1}^{n}\log\left(1-\left(1-H(x_i;\xi)^{\alpha}\right)^{\lambda}\right)\\&+(\lambda-1)\sum_{i=1}^{n}\log\left(1-H(x_i;\xi)^{\alpha}\right)+(\alpha-1)\sum_{i=1}^{n}\log\left[H(x_i;\xi)\right]+\sum_{i=1}^{n}log\left[1-\left[1-\left[1-\left[H(x_i;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{b}\right]\\&+(\theta-1)\sum_{i=1}^{n}log\left[1-\left[1-\left[1-\left[1-\left[H(x_i;\xi)\right]^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right]\end{aligned} \tag{40}$$

The components of score vector $U=(U_\alpha,U_\theta,U_b,U_\lambda)^T$ are given as

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[H(x_i;\xi)\right] + (\lambda-1)\sum_{i=1}^{n}\left[\frac{H(x_i;\xi)^\alpha \log\left[H(x_i;\xi)\right]}{1-H(x_i;\varphi)^\alpha}\right] + (b-1)\sum_{i=1}^{n}\left[\frac{\lambda\left[1-H(x_i;\xi)^\alpha\right]^{\lambda-1} H(x_i;\xi)^\alpha \log\left[H(x_i;\xi)\right]}{1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda}\right]$$

$$+(\theta-1)\sum_{i=1}^{n}\left[\frac{2\lambda b\left[1-\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]^b\right]\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^{b-1}\left[1-H(x_i;\xi)^\alpha\right]^{\lambda-1} H(x_i;\xi)^\alpha \log\left[H(x_i;\xi)\right]}{1-\left[1-\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^b\right]^2}\right]$$

$$+\sum_{i=1}^{n}\left[\frac{\lambda b\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^{b-1}\left[1-H(x_i;\xi)^\alpha\right]^{\lambda-1} H(x_i;\xi)^\alpha \log\left[H(x_i;\xi)\right]}{1-\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]^b}\right] = 0 \tag{41}$$

$$U_\lambda = (\theta-1)\sum_{i=1}^{n}\left[\frac{2b\left[1-\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]^b\right]\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^{b-1}\left[1-H(x_i;\xi)^\alpha\right]^\lambda \log\left[1-\left[H(x_i;\xi)\right]^\alpha\right]}{1-\left[1-\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^b\right]^2}\right]$$

$$+\frac{n}{\lambda} + \sum_{i=1}^{n} log\left[1-H(x_i;\xi)^\alpha\right] - (b-1)\sum_{i=1}^{n}\frac{\left[1-H(x_i;\xi)^\alpha\right]^\lambda log\left[1-H(x_i;\xi)^\alpha\right]}{1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda}$$

$$-\sum_{i=1}^{n}\left[\frac{\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^b \log\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]}{1-\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]^b}\right] = 0 \tag{42}$$

$$U_b = \frac{n}{b} + \sum_{i=1}^{n} log\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right] + \sum_{i=1}^{n}\left[\frac{\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^b log\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]}{1-\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]}\right] + (\theta-1)\sum_{i=1}^{n}\left[\frac{2\left[1-\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]\right]\left[1-\left[1-\left[H(x_{i;\xi})\right]^\alpha\right]^\lambda\right]^b log\left[1-\left[1-H(x_i;\xi)^\alpha\right]^\lambda\right]}{1-\left[1-\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^b\right]^2}\right] = 0 \tag{43}$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\left[1-\left[1-\left[1-\left[1-\left[H(x_i;\xi)\right]^\alpha\right]^\lambda\right]^b\right]^2\right] = 0 \tag{44}$$

The solution of the non-linear equations (41), (42), (43) and (44) cannot be obtained analytically so, we resolved to use iterative methods.

2.5 Sub-models of the ETLGenEx-G family of distributions

In this section, we provide some sub-models of the ETLGenEx-G family. The pdf of the ETLGenEx-G family will be most tractable when *f(x)* and *F(x)* have simple analytic expressions. These special models generalize some well-known distributions reported in the literature. Here, we provide two sub-models of this family corresponding to the baseline Exponential (Ex) and Weibull (W) distributions to show the flexibility of the new family.

2.5.1 The ETLGenEx exponential (ETLGenExEx) distribution

The parent exponential distribution has cdf and pdf given as

$$H(x;\beta) = 1 - e^{-\beta x} \tag{45}$$

$$h(x;\beta) = \beta e^{-\beta x} \tag{46}$$

The cdf and pdf of ETLGenExEx distribution are given by

$$F(x;\alpha,\beta,\lambda,\theta,b) = \left\{1 - \left[1 - \left[1 - \left[1 - \left[1 - e^{-\beta x}\right]^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right\}^{\theta} \tag{47}$$

$$f(x;\alpha,\beta,\lambda,\theta,b) = 2b\alpha\lambda\theta\beta e^{-\beta x}\left[1 - e^{-\beta x}\right]^{\alpha-1}\left[1 - \left[1 - e^{-\beta x}\right]^{\alpha}\right]^{\lambda-1}\left\{1 - \left[1 - \left[1 - e^{-\beta x}\right]^{\alpha}\right]^{\lambda}\right\}^{b-1}\left\{1 - \left[1 - \left[1 - \left[1 - e^{-\beta x}\right]^{\alpha}\right]^{\lambda}\right]^{b}\right\}^{\theta-1} \tag{48}$$

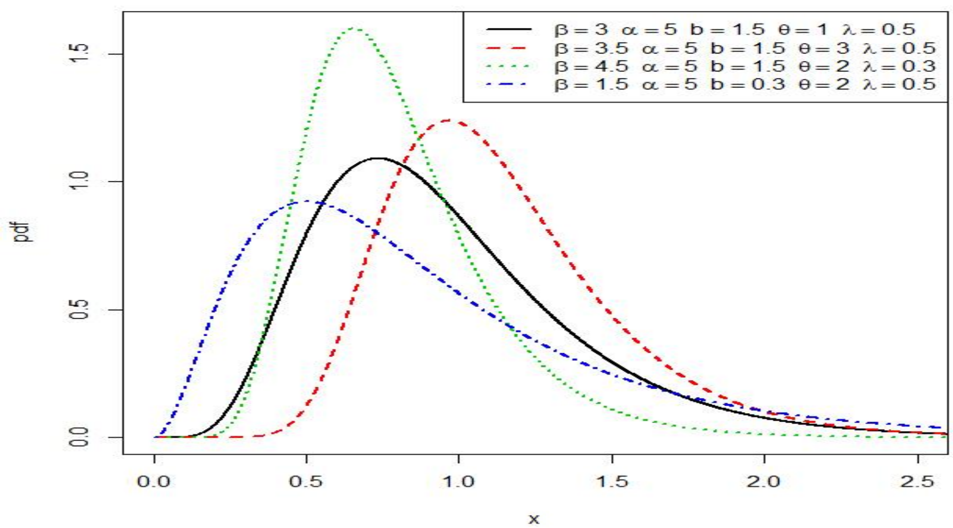

Figure 1. pdf plot of ETLGenExEx distribution with different parameter value

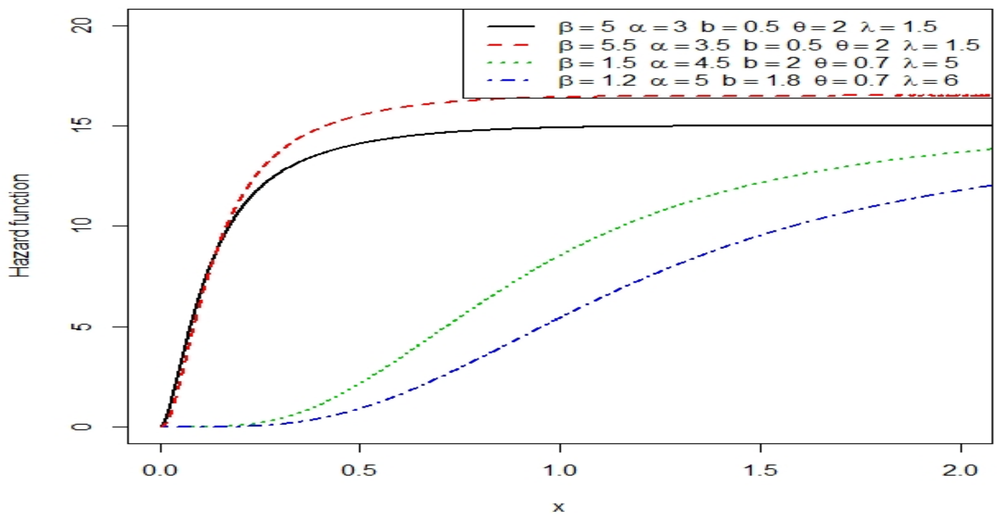

Figure 2. hrf plot of ETLGenExEx distribution with different parameter value

2.5.2 The ETLGenEx Weibull (ETLGenExW) distribution

The cdf and pdf of the Weibull distribution are given by

$$H\left(x;\beta,\varphi\right)=1-e^{-(\beta x)^{\varphi}} \tag{49}$$

$$h\left(x;\beta,\varphi\right)=\varphi\beta^{\varphi}x^{\varphi-1}e^{-(\beta x)^{\varphi}} \tag{50}$$

$x>0,\beta,\varphi>0$

Where β is the scale parameter and φ is the shape parameter.

$$F\left(x;\alpha,\beta,\lambda,\theta,\varphi\right)=\left\{1-\left[1-\left[1-\left[1-\left[1-e^{-(\beta x)^{\varphi}}\right]^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right\}^{\theta} \tag{51}$$

$$f\left(x;\alpha,\beta,\lambda,\theta,\varphi\right)=2b\alpha\beta\theta\lambda\varphi x^{\varphi-1}e^{-(\beta x)^{\varphi}}\left[1-e^{-\beta x}\right]^{\alpha-1}\left[1-\left[1-e^{-(\beta x)^{\varphi}}\right]^{\alpha}\right]^{\lambda-1}\left\{1-\left[1-\left[1-e^{-(\beta x)^{\varphi}}\right]^{\alpha}\right]^{\lambda}\right\}^{b-1}\left\{1-\left[1-\left[1-\left[1-e^{-(\beta x)^{\varphi}}\right]^{\alpha}\right]^{\lambda}\right]^{b}\right\}\left\{1-\left[1-\left[1-\left[1-\left[1-e^{-(\beta x)^{\varphi}}\right]^{\alpha}\right]^{\lambda}\right]^{b}\right]^{2}\right\}^{\theta-1} \tag{52}$$

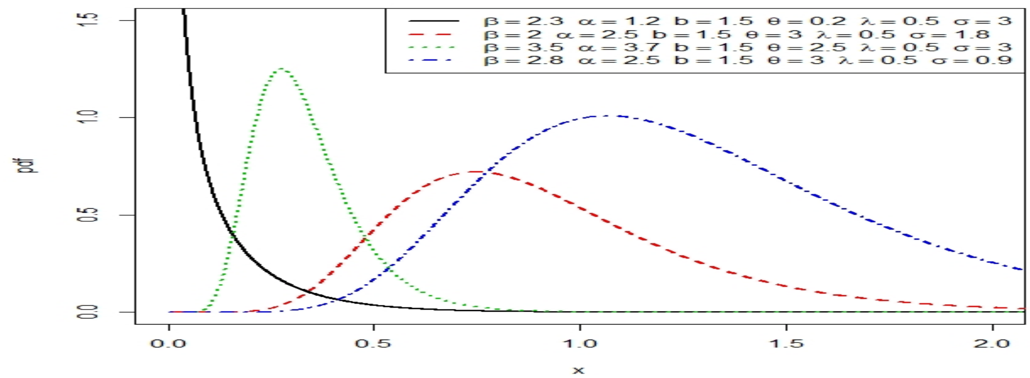

Figure 3. pdf plot of ETLGenExW distribution with different parameter value

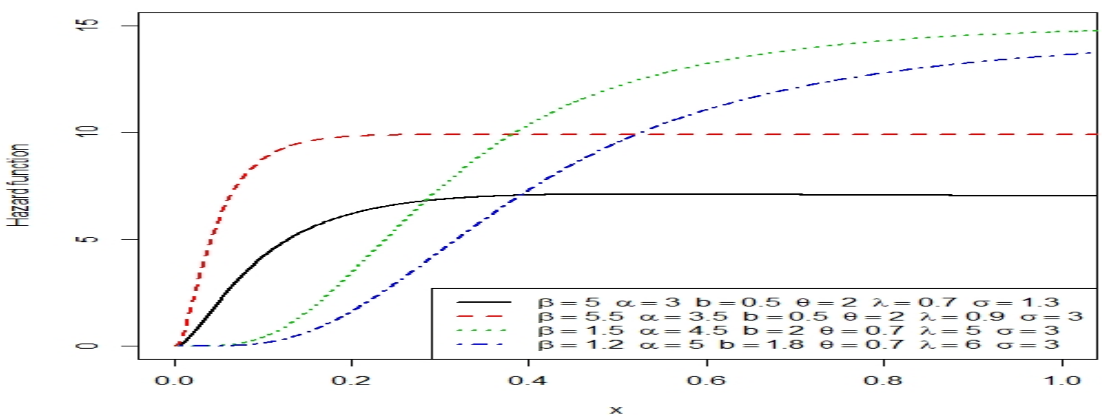

Figure 4. hrf plot of ETLGenExW distribution with different parameter value

From the figure 1, figure 2, figure 3 and figure 4, it can be seen that the model exhibits different shapes which can be increasing, decreasing, reversed J shape and can be positively or negatively skewed and also symmetric.

2.6 Application to real-life data

This section presents the real-life application of theE TLGenEx-G family of distributions to a real-life data set by using the ETLGenExEx distribution as one of the sub-models and for illustrative purposes also present a comparative study with the fits of Extended generalized inverse exponential (EgenIEx) distribution by Ibrahim et al., (2020c), Topp Leone exponential (TLEx) distribution by Al-Shomrani et al., (2016), exponentiated exponential (ExEx) distribution by Gupta and Kundu (1999), exponentiated Lomax (ExLx) distribution by Salem (2014), inverse exponential (IEx) distribution by Keller and Kamath (1982) and exponential (Ex) distribution. This application proved empirically the flexibility of the new distribution in modeling real-life data. In order to compare the fits of the ETLGenExEx distribution with other competing distributions, we consider the Cramér-Von-Mises (W), the Anderson-Darling (A), Kolmogorov Smirnov statistics and probability value (PV). Also, to make the comparism valid, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) was used. All the computations are performed using the *AdequacyModel* package in *R software*

The data set used in this paper represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The data set was given by Gross and Clark (1975) and has also been used by Shanker (2015) to assess the flexibility of Exponential distribution and Lindley distribution.

The data set consists of twenty (20) observations and it is as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2

Table 1. Goodness of fit statistics from relief times (in minutes) of patients receiving an analgesic.

Model	Estimates	-LL	AIC	BIC	W	A	KS	PV
ETLGenExEx	$\hat{\alpha} = 4.8070$ $\hat{\beta} = 0.8105$ $\hat{\theta} = 3.6225$ $\hat{\lambda} = 1.8022$ $\hat{\delta} = 0.6391$	17.2470	44.4940	49.4727	0.0837	0.4956	0.1221	0.9268
EGenIEx	$\hat{\alpha} = 2.6453$ $\hat{\beta} = 1.2492$ $\hat{\theta} = 1.6218$ $\hat{\lambda} = 1.8058$	19.9030	47.8060	51.7890	0.0471	0.2742	0.1703	0.6078
ExLx	$\hat{\alpha} = 4.5360$ $\hat{\beta} = 0.3178$ $\hat{\theta} = 4.9008$	24.0688	54.1376	57.1248	0.0683	0.4023	0.2338	0.2242
ExEx	$\hat{\beta} = 0.6675$ $\hat{\theta} = 1.9923$	27.1091	58.2182	60.2096	0.0997	0.5904	0.2720	0.1038
TLEx	$\hat{\beta} = 0.3309$ $\hat{\theta} = 1.9103$	27.3691	58.7383	60.7297	0.1001	0.5930	0.2837	0.0799
IEx	$\hat{\beta} = 1.7244$	32.6687	67.3373	68.3331	0.0489	0.2852	0.3872	0.0050
Ex	$\hat{\beta} = 0.5256$	32.8371	67.6742	68.6699	0.1054	0.6243	0.4392	0.0009

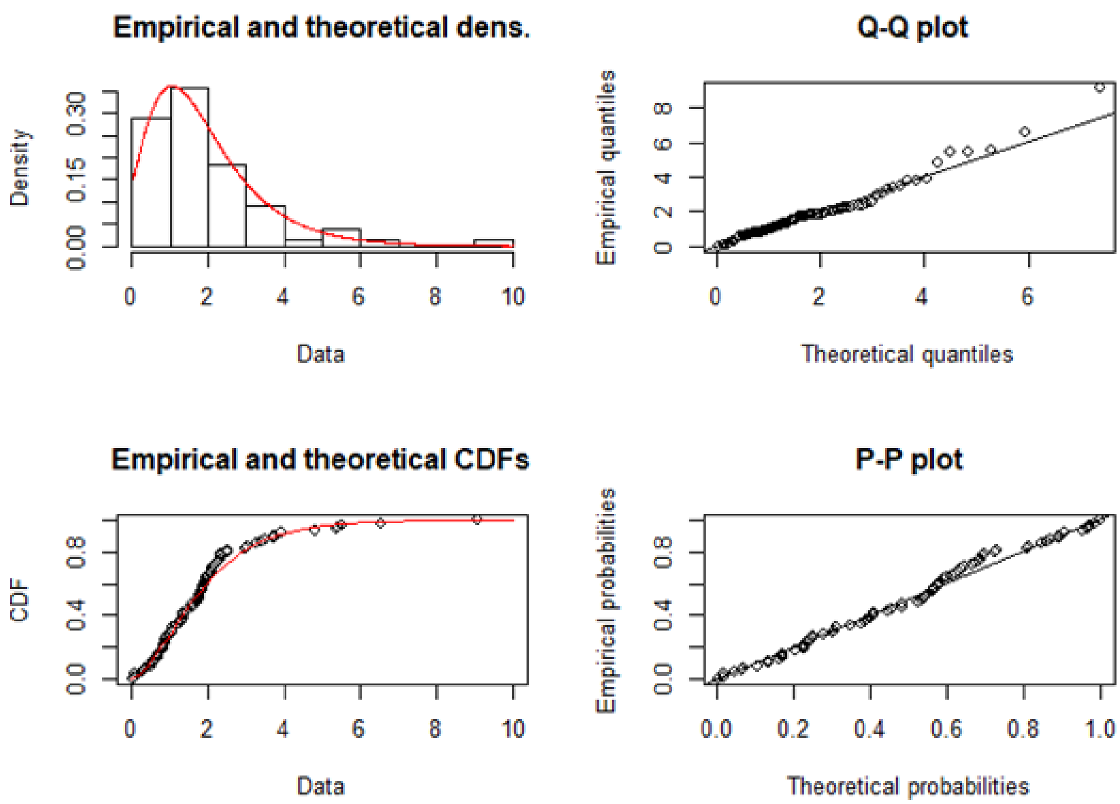

Figure 5. Fitted pdf, cdf,Q-Q plot and P-P plot of the data set

3. Results and Discussion

The methodology developed by Alzaatreh et al., (2013) was used to obtain a new family of distributions which is an extension of the family proposed by Rasheed (2020). The new family developed was applied to two distributions such as exponential and Weibull distributions. The pdfs of the two proposed distributions are; the ETLGenExEx and ETLGenExW distributions. The shapes of the ETLGenExEx distribution are presented in Figure 1 and Figure 2. From the figures, it can be seen that the distribution can be used to model skewed data. Also, the shapes of the ETLGenExW are presented in Figure 3 and Figure 4. From the figures, it can also be seen that the distribution can be used to model left skewed, right skewed or symmetric data sets.

4. Conclusion

As a result of the study carried out, a new family of distributions was derived. Some mathematical properties of the new family were studied and presented such as moments, moment generating function, quantile function, hazard function, survival function, odd function, cumulative hazard function, Renyi entropy and order statistics. The method of maximum likelihood estimate was used to estimate the parameters of the model. The proposed ETLGenExEx distribution was applied to a real life data set to test the flexibility of the model. From the result shown in table 1, it can be seen that

ETLGenExEx distribution has the lowest AIC and BIC and also has the highest P-Value which makes the proposed model fit better than other competing distributions considered in the study with respect to the data set used.

5. Acknowledgements

The authors expressed their profound gratitude to the editorial crew and anonymous reviewers for their constructive comments and suggestions which led to remarkable improvement of the paper.

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